

Application of Darboux Transformation to solve Multisoliton Solution on Non-linear Schrödinger Equation

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Darboux transformation is one of the methods used in solving nonlinear evolution equation. Basically, the Darboux transformation is a linear algebra formulation of the solutions of the Zakharov-Shabat system of equations associated with the nonlinear evolution equation. In this work, the evolution of monochromatic electromagnetic wave in a nonlinear-dispersive optical medium is considered. Using the Darboux transformation, explicit multisoliton solutions (one to three soliton solutions) are obtained from a trivial initial solution.

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I. INTRODUCTION

In 1882, Gaston Darboux introduced a method to solve Sturm-Liouville differential equation, which is called Darboux transformation afterwards[1]. Remarkably, this transformation can solve not only Sturm-Liouville differential equation (another form of time-independent Schrödinger equation), but also for the other forms of linear and non-linear differential equations, such as Korteweg-de Vries, Kadomtsev-Petviashvili, Sine-Gordon, and Non-linear Schrödinger equations. Following exposition explains about Darboux transformation and its generalized form called Crum's theorem on Sturm-Liouville differential equation.

II. DARBOUX TRANSFORMATION AND CRUM THEOREM

Consider following Sturm-Liouville differential equation

$$-\psi_{xx} + u\psi = \lambda\psi \quad (1)$$

where u is function of x and λ is a constant. In the Schrödinger equation, u represent a potential. Darboux transformation is defined as follow[1]

$$\begin{aligned} \psi[1] &= \left(\frac{d}{dx} - \sigma_1 \right) \psi = \psi_x - \frac{\psi_{1x}}{\psi_1} \psi \\ &= \frac{\psi_x \psi_1 - \psi_{1x} \psi}{\psi_1} = \frac{W(\psi_1, \psi)}{W(\psi_1)} \end{aligned} \quad (2)$$

$W(\psi, \psi[1])$ expresses Wronskian determinant as below

$$W(\psi_1, \psi_1, \dots, \psi_N) = \begin{vmatrix} \psi_1 & \psi_2 & \dots & \psi_N \\ \psi_1^{(1)} & \psi_2^{(1)} & \dots & \psi_N^{(1)} \\ \dots & \dots & \dots & \dots \\ \psi_1^{(n-1)} & \psi_2^{(n-1)} & \dots & \psi_N^{(n-1)} \end{vmatrix}. \quad (3)$$

where $\psi^{(n)}$ means ψ derivative for n -times and ψ_1 is the solution of ψ for $\lambda = \lambda_1$. If ψ is a solution, $\psi[1]$ is the so-

lution for the following Sturm-Liouville differential equation then

$$-\psi_{xx}[1] + u[1]\psi[1] = \lambda\psi[1] \quad (4)$$

where $u[1]$ is a new function of u which has been transformed. It will be shown below that Darboux transformation acting on $\psi[1]$ influences the potential u , if $\psi[1]$ is invariant over Eq. (4).

From Eq.(2), one can obtain following relation

$$-\psi_{xx}[1] = -\psi_{xxx} + \sigma_{1xx}\psi + 2\sigma_{1x}\psi_x + \sigma_1\psi_{xx} \quad (5)$$

Substitution of Eq.(2) and Eq.(5) into Eq.(4) results

$$\begin{aligned} -\psi_{xxx} + \sigma_{1xx}\psi + 2\sigma_{1x}\psi_x + \sigma_1\psi_{xx} + u[1](\psi_x - \sigma_1\psi) \\ = \lambda(\psi_x - \sigma_1\psi) \end{aligned} \quad (6)$$

Using Eq.(1) to substitute ψ_{xx} results

$$\begin{aligned} (u[1] - u + 2\sigma_{1x})\psi_x + \\ (-u_x + \sigma_{1xx} + \sigma_1 u - \sigma_1 u[1])\psi = 0 \end{aligned} \quad (7)$$

It is clear from Eq.(7), the following relations can be obtained

$$u[1] = u - 2\sigma_{1x} \quad (8)$$

$$\sigma_{1xx} - u_x + 2\sigma_1\sigma_{1x} = 0 \quad (9)$$

The Eq.(5) shows that function u is also under the transformation due to ψ in Eq.(1) under Darboux transformation. In other words, Sturm-Liouville equation in Eq.(1) is covariant under Darboux transformation action

$$\psi \rightarrow \psi[1] \text{ and } u \rightarrow u[1]$$

Interestingly, Darboux transformation can be recursively applied to Sturm-Liouville equation solution and consequently the potential is under transformation to ensure that the solution belongs to the equation. Below will be discussed the consequence if the solution in Eq.(1) is acted by Darboux transformation twice.

$$\psi[2] = \left(\frac{d}{dx} - \frac{\psi_{2x}[1]}{\psi_2[1]} \right) \left(\frac{d}{dx} - \frac{\psi_{1x}}{\psi_1} \right) \psi \quad (10)$$

where

$$\psi[2] = \psi_{2x} - \frac{\psi_{1x}}{\psi_1} \psi_2 \quad (11)$$

Therefore

$$\begin{aligned} \psi[2] &= \left(\frac{d}{dx} - \frac{\left(\psi_{2x} - \frac{\psi_{1x}}{\psi_1} \psi_2 \right)_x}{\psi_{2x} - \frac{\psi_{1x}}{\psi_1} \psi_2} \right) \left(\frac{d\psi}{dx} - \frac{\psi_{1x}\psi}{\psi_1} \right) = \frac{d}{dx} \left(\frac{d\psi}{dx} - \frac{\psi_{1x}\psi}{\psi_1} \right) - \frac{\left(\psi_{2x} - \frac{\psi_{1x}}{\psi_1} \psi_2 \right)_x}{\psi_{2x} - \frac{\psi_{1x}}{\psi_1} \psi_2} \left(\frac{d\psi}{dx} - \frac{\psi_{1x}\psi}{\psi_1} \right) \\ &= \frac{\psi_1 (\psi_{xx}\psi_{2x} - \psi_{2xx}\psi_x) - \psi_2 (\psi_{xx}\psi_{1x} - \psi_{1xx}\psi_x) + -\psi (\psi_{2xx}\psi_{1x} - \psi_{1xx}\psi_{2x})}{W(\psi_1, \psi_2)} = \frac{\begin{vmatrix} \psi_1 & \psi_2 & \psi \\ \psi_{1x} & \psi_{2x} & \psi_x \\ \psi_{1xx} & \psi_{2xx} & \psi_{xx} \end{vmatrix}}{W(\psi_1, \psi_2)} \quad (12) \end{aligned}$$

From the definition $\psi[1]$ and $\psi[2]$, $u[1]$, and $u[2]$, Crum proposed general form of Darboux transformation in the following form

$$\psi[N] = \frac{W(\psi_1, \dots, \psi_N, \psi)}{W(\psi_1, \dots, \psi_N)} \quad (13a)$$

$$u[N] = u - 2 \frac{\partial^2}{\partial x^2} \ln W(\psi_1, \dots, \psi_N) \quad (13b)$$

where $\psi[N]$ satisfies the equation

$$-\psi_{xx}[N] + u[N]\psi[N] = \lambda\psi[N] \quad (14)$$

The Crum's theorem is based on the fact that Darboux transformation acting N -times on a function can be written

$$\psi[N] = D[N]\psi = \psi^{(N)} + s_1\psi^{(N-1)} + \dots + s_N\psi. \quad (15)$$

Ansatz of Crum's theorem is that s_N is obtained from the following relation

$$\sum_{k=1}^N s_k \psi_j^{N-k} = -\psi_j^{(N)} \quad (16)$$

where $j=1, 2, \dots, N$. The equation (16) can be explained and arranged in matrix representation by following

$$\begin{pmatrix} \psi_1^{(N-1)} & \dots & \dots & \psi_1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \psi_N^{(N-1)} & \dots & \dots & \psi_N \end{pmatrix} \begin{pmatrix} s_1 \\ \dots \\ \dots \\ s_N \end{pmatrix} = \begin{pmatrix} -\psi_1^{(N)} \\ \dots \\ \dots \\ -\psi_N^{(N)} \end{pmatrix} \quad (17)$$

By the use of Cramer's rule, one can obtain s_j for $j = 1,$

\dots, N . For instance, s_1

$$s_1 = \frac{\begin{vmatrix} -\psi_1^{(N)} & \psi_1^{(N-1)} & \dots & \dots & \psi_1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -\psi_N^{(N)} & \psi_N^{(N-1)} & \dots & \dots & \psi_N \end{vmatrix}}{\begin{vmatrix} \psi_1^{(N-1)} & \dots & \dots & \psi_1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \psi_N^{(N-1)} & \dots & \dots & \psi_N \end{vmatrix}}$$

For example, the implementation of Crum's theorem as given above will be proven below for $N=2$. The equation (16) for $N=2$ can be explained by following

$$s_1\psi_{1x} + s_2\psi_1 = -\psi_{1xx} \quad (18a)$$

$$s_1\psi_{2x} + s_2\psi_2 = -\psi_{2xx} \quad (18b)$$

By the use of Cramer's rule, one can determine s_1 and s_2 as below

$$s_1 = \frac{\begin{vmatrix} -\psi_{1xx} & \psi_1 \\ -\psi_{2xx} & \psi_2 \end{vmatrix}}{\begin{vmatrix} \psi_{1x} & \psi_1 \\ \psi_{2x} & \psi_2 \end{vmatrix}} = -\frac{\begin{vmatrix} \psi_1 & \psi_{1xx} \\ \psi_2 & \psi_{2xx} \end{vmatrix}}{\begin{vmatrix} \psi_1 & \psi_2 \\ \psi_{1x} & \psi_{2x} \end{vmatrix}} \quad (19a)$$

$$s_1 = \frac{\begin{vmatrix} \psi_{1x} & -\psi_{1xx} \\ \psi_{2x} & -\psi_{2xx} \end{vmatrix}}{\begin{vmatrix} \psi_{1x} & \psi_1 \\ \psi_{2x} & \psi_2 \end{vmatrix}} = -\frac{\begin{vmatrix} \psi_{1x} & \psi_{1xx} \\ \psi_{2x} & \psi_{2xx} \end{vmatrix}}{\begin{vmatrix} \psi_1 & \psi_2 \\ \psi_{1x} & \psi_{2x} \end{vmatrix}} \quad (19b)$$

Darboux operator in Eq.(15) for $N = 2$ can be described

$$\psi[2] = D[2]\psi = \psi_{xx} + s_1\psi_x + s_2\psi. \quad (20)$$

Whilst, substitution Eq.(19) into Eq.(20) yields

$$\psi[2] = D[2]\psi = \frac{W(\psi_1, \psi_2, \psi)}{W(\psi_1, \psi_2)}$$

III. DARBOUX TRANSFORMATION ON NONLINEAR SCHRÖDINGER EQUATION

From Zakharov-Shabat system, one can derive several nonlinear differential equations, such as nonlinear Schrödinger and Sine-Gordon equations [1]. In 1982, Bobenko and Salle can apply Darboux transformation on generalized Zakharov-Shabat system[2]. In this section, the Darboux transformation application for first order linear Zakharov-Shabat system will be explained, followed by its application for nonlinear Schrödinger equation.

A. Darboux Transformation for First Order Linear System

Consider the following system:

$$\psi_x = -i\lambda\psi + iq\phi. \quad (21a)$$

$$\phi_x = -\lambda\phi + ir\psi. \quad (21b)$$

These differential equations can be assembled into the following matrix:

$$\Psi_x = J\Psi\Lambda + U\Psi \quad (22)$$

where

$$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \phi_1 & \phi_2 \end{pmatrix}$$

The solution for Eq.(21) is $\begin{pmatrix} \psi_k \\ \phi_k \end{pmatrix}$ where $k=1, 2$. Darboux transformation acting on Eq.(22) is defined by following

$$\Psi \rightarrow \Psi[1] = D[1]\Psi = \Psi\Lambda + S_1\Psi = \Psi\Lambda - (\Psi_1\Lambda_1\Psi_1^{-1}) \quad (23)$$

where Ψ_1 is the solution obtained for Eq.(22) with $\Lambda = \Lambda_1$. The equation (22) is called covariant under Darboux transformation if the potential is transformed

$$U[1] = U + [S_1, J]. \quad (24)$$

Therefore, $\Psi[1]$ is the solution of following differential equation

$$\Psi[1] = J\Psi[1]\Lambda + U[1]\Psi[1]. \quad (25)$$

Acting N -times Darboux transformation on Eq.(22), it means following this rule

$$D[N]\Psi = \Psi[N] = \Psi\Lambda^N + S_1\Psi\Lambda^{N-1} + \dots + S_{N-1}\Psi\Lambda + S_N\Psi. \quad (26)$$

Whilst, generalized form of the potential transforms

$$U[N] = U + [S_N, J]. \quad (27)$$

From Crum's theorem, S_i can be obtained from the following condition

$$\Psi[N]_{\Lambda=\Lambda_k}, \Psi=\Psi_k = 0. \quad (28)$$

Because k is running until N , the set of equations in Eq.(28) can be rewritten in the following form then

$$\begin{array}{ccccccc} S_1\Psi_1\Lambda_1^{N-1} + & S_2\Psi_1\Lambda_1^{N-2} + & \dots & + S_N\Psi_1 & = & -\Psi_1\Lambda_1^N. \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ S_1\Psi_N\Lambda_1^{N-1} + & S_2\Psi_N\Lambda_1^{N-2} + & \dots & + S_N\Psi_N & = & -\Psi_N\Lambda_1^N. \end{array} \quad (29)$$

If the equations in (29) are assembled into matrix form

$$\begin{pmatrix} S_1 & \dots & S_N \end{pmatrix} \begin{pmatrix} \Psi_1\Lambda_1^{N-1} & \dots & \Psi_1 \\ \dots & \dots & \dots \\ \Psi_N\Lambda_N^{N-1} & \dots & \Psi_N \end{pmatrix} = \begin{pmatrix} -\Psi_1\Lambda_1^N & \dots & -\Psi_N\Lambda_N^N \end{pmatrix}. \quad (30)$$

Where

$$S_N = \begin{pmatrix} S_N^{11} & S_N^{12} \\ S_N^{21} & S_N^{22} \end{pmatrix}, \quad \Psi_N = \begin{pmatrix} \psi_{2N-1} & \psi_{2N} \\ \phi_{2N-1} & \phi_{2N} \end{pmatrix},$$

$$\Lambda_N = \begin{pmatrix} \lambda_{2N-1} & 0 \\ 0 & \lambda_{2N} \end{pmatrix}$$

S_i , for $i=1, \dots, N$, can be determined by the use of Cramer's rule. Therefore, for S_1 , one can obtain

$$S_1 = \frac{1}{\Delta} \begin{pmatrix} S_1^{11} & S_1^{12} \\ S_1^{21} & S_1^{22} \end{pmatrix} \quad (31)$$

where

$$\begin{aligned} \Delta &= \det(\lambda_i^{N-1}\psi_i, \lambda_i^{N-1}\phi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\phi_i, \dots, \psi_i, \phi_i), \\ S_1^{11} &= \det(\lambda_i^N\psi_i, \lambda_i^{N-1}\phi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\phi_i, \dots, \psi_i, \phi_i), \\ S_1^{22} &= \det(\lambda_i^{N-1}\psi_i, \lambda_i^N\phi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\phi_i, \dots, \psi_i, \phi_i), \\ S_1^{12} &= \det(\lambda_i^{N-1}\psi_i, \lambda_i^N\psi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\phi_i, \dots, \psi_i, \phi_i), \\ S_1^{21} &= \det(\lambda_i^N\phi_i, \lambda_i^{N-1}\phi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\phi_i, \dots, \psi_i, \phi_i). \end{aligned}$$

Where $i=1, \dots, 2N$ and also using following definition

$$\det(a_i, b_i, \dots, f_i) = \det \begin{pmatrix} a_1 & b_1 & \dots & f_1 \\ a_1 & b_1 & \dots & f_1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Consequently, N -times Darboux transformation on Eq.(27), $q[N]$ and $r[N]$ are expressed

$$q[N] = q + 2\frac{S_1^{12}}{\Delta} \quad (32a)$$

$$r[N] = r + 2\frac{S_1^{21}}{\Delta} \quad (32b)$$

The application of the entire calculation in this section for obtaining multisoliton of nonlinear Schrödinger equation will be explained in the following section.

B. Darboux Transformation and Crum's Theorem on Nonlinear Schrödinger Equation

Below, the derivation of nonlinear Schrödinger equation from generalized linear Zakharov-Shabat equation will be given, followed by the explanation of Darboux operator implementation on its solution then.

Consider following two linear equations

$$\Psi_{kx} = J\Psi_k\Lambda_k + U\Psi_k \quad (33a)$$

$$\Psi_{kt} = 2J\Psi_k\Lambda_k^2 + 2U\Psi_k\Lambda_k + (JU^2 - JU_x)\Psi_k. \quad (33b)$$

Where

$$U = \begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_{2k-1} & 0 \\ 0 & \lambda_{2k-1} \end{pmatrix}, \quad (34)$$

$$\Psi_k = \begin{pmatrix} \psi_{2k-1} & \psi_{2k} \\ \phi_{2k-1} & \phi_{2k} \end{pmatrix}.$$

From the integrability condition, $\Psi_{xt} = \Psi_{tx}$, it yields

$$(J^2U^2 - J^2U_x - 2U_x + JU_xJ - JU_x^2J)\Psi\Lambda + (U_t + UJU^2 - UJU_x - 2JU_xU - JU_x^2U + JU_{xx} + JU_xU)\Psi = 0. \quad (35)$$

If Eq.(35) is assembled in matrix form

$$\begin{pmatrix} qr - q_xr_x & 0 \\ 0 & qr - q_xr_x \end{pmatrix} \Psi\Lambda + \begin{pmatrix} -iqr_x + 2iqr_x - iqr_x & iq_t - q^2r - qq_xr_x - q_{xx} \\ ir_t + qr^2 + q_xr_xr + r_{xx} & iq_r - 2iq_r + iq_r \end{pmatrix} \Psi = 0 \quad (36)$$

For $q = r^*$ and $r = q^*$, this consistency equation (36) gives

$$ir_t + 2r^*r^2 + r_{xx} = 0, \quad (37a)$$

$$iq_t - 2q^*q^2 - q_{xx} = 0. \quad (37b)$$

It is clear that the equation (37) yields nonlinear

Schrödinger equation. Furthermore, it will be shown the consequence if the two linear equations in (33) are transformed under Darboux transformation (23), $\Psi_x[1] = J\Psi[1]\Lambda + U[1]\Psi[1]$ or $(\Psi\Lambda - \sigma\Psi)_x = J[1](\Psi\Lambda - S_1\Psi)\Lambda + U[1](\Psi\Lambda - S_1\Psi)$. The substitution of the equation (33) results

$$(J - J[1])\Psi\Lambda^2 + (U - S_1J + J[1]S_1 - U[1])\Psi\Lambda + (S_{1x} + S_1U - U[1]S_1)\Psi = 0 \quad (38)$$

From the equation (38) of each coefficient, one can determine

$$J = J[1] \quad (39a)$$

$$U[1] = U + J[1]S_1 - S_1J \Rightarrow U[1] = U + [J, S_1] \quad (39b)$$

$$S_{1x} + U - U[1]S_1 = 0 \quad (39c)$$

where $J[1]$ and $U[1]$ are respectively J and U in the equation (33) after Darboux transformation has been

accomplished.

C. Darboux Transformation Solution of Nonlinear Schrödinger Equation for Trivial Initial Solution

In this section, we will examine the consequence on nonlinear Schrödinger equation if its initial solution is trivial. For the case $r = q = 0$, the equation (33) will be in the form

$$\begin{pmatrix} \psi_{2k-1} & \psi_{2k} \\ \phi_{2k-1} & \phi_{2k} \end{pmatrix}_x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \psi_{2k-1} & \psi_{2k} \\ \phi_{2k-1} & \phi_{2k} \end{pmatrix} \begin{pmatrix} \lambda_{2k-1} & 0 \\ 0 & \lambda_{2k} \end{pmatrix}. \quad (40a)$$

$$\begin{pmatrix} \psi_{2k-1} & \psi_{2k} \\ \phi_{2k-1} & \phi_{2k} \end{pmatrix}_t = 2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \psi_{2k-1} & \psi_{2k} \\ \phi_{2k-1} & \phi_{2k} \end{pmatrix} \begin{pmatrix} \lambda_{2k-1} & 0 \\ 0 & \lambda_{2k} \end{pmatrix}^2. \quad (40b)$$

The equations in (40) yield the following solution

$$\psi_{2k-1} = A_{2k-1} e^{i(\lambda_{2k-1}x + 2\lambda_{2k-1}^2t)}; \quad \psi_{2k} = A_{2k} e^{i(\lambda_{2k}x + 2\lambda_{2k}^2t)} \quad (41a)$$

$$\phi_{2k-1} = B_{2k-1} e^{-i(\lambda_{2k-1}x + 2\lambda_{2k-1}^2t)}; \quad \phi_{2k} = B_{2k} e^{-i(\lambda_{2k}x + 2\lambda_{2k}^2t)} \quad (41b)$$

The solutions in (41) show that from the trivial solution in (33) by using Darboux transformation can create multisoliton as mentioned in the following sections.

IV. MULTISOLITON OF NONLINEAR SCHRÖDINGER EQUATION USING DARBOUX TRANSFORMATION

In this chapter, the calculation of multisoliton from trivial solution using Darboux transformation will be explained. In the previous chapter has been shown that by choosing appropriate linear Zakharov-Shabat system and also from consistency condition, two types of nonlinear Schrödinger equation can be obtained, where either of them is

$$ir_t + r_{xx} + 2|r|^2 r = 0. \quad (42)$$

For N -times Darboux transformations, the equation is matrix element of $U[N]$, i.e. $(U[N])_{21}$, which is in the following form

$$i(r[N])_t + (r)_{xx} + 2|r[N]|^2 r[N] = 0. \quad (43)$$

Below, we will describe how to generate multisoliton solution from trivial solution for 3-times Darboux transformation action.

A. The Calculation of 1-Soliton of Nonlinear Schrödinger Equation using Darboux Transformation

For the case $N = 1$, according to the previous explanation, from the equation (31) yields

$$S_1^{21} = \begin{vmatrix} \lambda_1 \phi_1 & \lambda_2 \phi_2 \\ \phi_1 & \phi_2 \end{vmatrix} = (\lambda_1 - \lambda_2) \phi_1 \phi_2 = (\lambda_1 - \lambda_1^*) \psi_1^* \phi_1. \quad (44)$$

Furthermore, by choosing $\psi_2 = -\phi_1^*$, it is resulted

$$\begin{aligned} \Delta &= \begin{vmatrix} \psi_1 & \psi_2 \\ \phi_1 & \phi_2 \end{vmatrix} = (\psi_1 \phi_2 - \psi_2 \phi_1) = (\psi_1 \psi_1^* - \phi_1 (-\phi_1^*)) \\ &= |\psi_1|^2 + |\phi_1|^2 \end{aligned} \quad (45)$$

Consequently, the initial trivial solution of equation (32) gives

$$r[1] = -2 \frac{S_1^{21}}{\Delta} = -2 \frac{(\lambda_1 - \lambda_1^*) \psi_1^* \phi_1}{|\psi_1|^2 + |\phi_1|^2}. \quad (46)$$

Let us define the following variable

$$\theta = \lambda_1 x + 2t\lambda_1^2 = \text{Re}(\theta) + i\text{Im}(\theta). \quad (47)$$

Substitution the equation (47) into (41) for the case $k = 1$, followed by substitution into (46) afterwards. One can determine

$$r[1] = -2 \frac{(2i\text{Im}(\lambda_1)) A_1 B_1 e^{(-2i\text{Im}(\theta))}}{(|A_1|^2 e^{2(\text{Re}(\theta))} + |B_1|^2 e^{-2(\text{Re}(\theta))})} \quad (48)$$

Furhtermore, let us define the following constants

$$A_1 = e^{\frac{\Phi_2 + i\Phi_1}{2}}, \quad (49a)$$

$$B_1 = e^{\frac{-\Phi_2 + i\Phi_1}{2}}. \quad (49b)$$

Substitution the equation (49) into (47), one can obtain the following solution

$$r[1] = -\frac{(2i\text{Im}(\lambda_1)) e^{(-2i\text{Im}(\theta) + \Phi_1)}}{\cosh(2(\text{Re}(\theta) + \Phi_2))} \quad (50)$$

The equation (50) is the solution of 1-soliton of nonlinear Schrödinger equation which has the form (42). From (50), it can be shown that imaginary component of λ_1 related to wave amplitude, while its real component represents the wave velocity.

B. The Calculation of 2-Soliton of Nonlinear Schrödinger Equation using Darboux Transformation

For 2-soliton, according to the Crum's theorem

$$r[2] = -2 \frac{S_1^{21}}{\Delta}. \quad (51)$$

Where

$$S_1^{21} = \begin{vmatrix} \lambda_1^2 \phi_1 & \lambda_2^2 \phi_2 & \lambda_3^2 \phi_3 & \lambda_4^2 \phi_4 \\ \lambda_1 \phi_1 & \lambda_2 & \lambda_3 \phi_3 & \lambda_4 \phi_4 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix} \quad (52a)$$

$$\Delta = \begin{vmatrix} \lambda_1 \psi_1 & \lambda_2 \psi_2 & \lambda_3 \psi_3 & \lambda_4 \psi_4 \\ \lambda_1 \phi_1 & \lambda_2 & \lambda_3 \phi_3 & \lambda_4 \phi_4 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix} \quad (52b)$$

Furthermore, by substituting (52) into (51), one can obtain

$$r[2] = -2 \frac{\begin{vmatrix} \lambda_1^2 \phi_1 & \lambda_2^2 \phi_2 & \lambda_3^2 \phi_3 & \lambda_4^2 \phi_4 \\ \lambda_1 \phi_1 & \lambda_2 & \lambda_3 \phi_3 & \lambda_4 \phi_4 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix}}{\begin{vmatrix} \lambda_1 \psi_1 & \lambda_2 \psi_2 & \lambda_3 \psi_3 & \lambda_4 \psi_4 \\ \lambda_1 \phi_1 & \lambda_2 & \lambda_3 \phi_3 & \lambda_4 \phi_4 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{vmatrix}} \quad (53)$$

By defining the following parameters

$$(\text{Re}(\lambda_1) - \text{Re}(\lambda_3)) + i(\text{Im}(\lambda_1) - \text{Im}(\lambda_3)) = \rho^* \quad (54a)$$

$$-((\text{Re}(\lambda_1) - \text{Re}(\lambda_3))) - i(\text{Im}(\lambda_1) - \text{Im}(\lambda_3)) = -\rho \quad (54b)$$

$$(\text{Re}(\lambda_1) - \text{Re}(\lambda_3)) + i(\text{Im}(\lambda_1) + \text{Im}(\lambda_3)) = \beta \quad (54c)$$

$$(\text{Re}(\lambda_1) - \text{Re}(\lambda_3)) - i(\text{Im}(\lambda_1) + \text{Im}(\lambda_3)) = \beta^* \quad (54d)$$

Where

$$\theta = \lambda_1 x + 2t\lambda_1^2 = \text{Re}(\theta) + i\text{Im}(\theta) \quad (54e)$$

$$\alpha = \lambda_3 x + 2t\lambda_3^2 = \text{Re}(\alpha) + i\text{Im}(\alpha) \quad (54f)$$

One can obtain the solution for 2-soliton as stated below

$$r[2] = 4i \frac{\text{Im}(\lambda_1) e^{2i\text{Im}(\theta)} (\rho^* \beta e^{-2\text{Re}(\alpha)} + \rho \beta^* e^{-2\text{Re}(\alpha)}) - \text{Im}(\lambda_3) e^{2i\text{Im}(\alpha)} (\rho^* \beta^* e^{-2\text{Re}(\theta)} + \rho \beta e^{-2\text{Re}(\theta)})}{|\rho|^2 (\cosh(2(\text{Re}(\theta) + \text{Re}(\alpha)))) + |\beta|^2 (\cosh(2(\text{Re}(\theta) - \text{Re}(\alpha)))) - 4\text{Im}(\lambda_1) \text{Im}(\lambda_3) (\cosh(2(\text{Im}(\theta) + \text{Im}(\alpha))))} \quad (55)$$

The equation (55) is obtained by choosing $\phi_2 = -\psi_1^*$ and $\psi_4 = -\phi_3^*$.

C. The Calculation of 3-Soliton of Nonlinear Schrödinger Equation using Darboux Transformation

3-soliton can be generated by similar manner as obtaining 2-soliton. First, determinant form of S_1^{21} and Δ which are in the following forms

$$S_1^{21} = \begin{vmatrix} \lambda_1^3 \phi_1 & \lambda_2^3 \phi_2 & \lambda_3^3 \phi_3 & \lambda_4^3 \phi_4 & \lambda_5^3 \phi_5 & \lambda_6^3 \phi_6 \\ \lambda_1^2 \phi_1 & \lambda_2^2 \phi_2 & \lambda_3^2 \phi_3 & \lambda_4^2 \phi_4 & \lambda_5^2 \phi_5 & \lambda_6^2 \phi_6 \\ \lambda_1 \psi_1 & \lambda_2 \psi_2 & \lambda_3 \psi_3 & \lambda_4 \psi_4 & \lambda_5 \psi_5 & \lambda_6 \psi_6 \\ \lambda_1 \phi_1 & \lambda_2 \phi_2 & \lambda_3 \phi_3 & \lambda_4 \phi_4 & \lambda_5 \phi_5 & \lambda_6 \phi_6 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 \end{vmatrix} \quad (56a)$$

$$\Delta = \begin{vmatrix} \lambda_1^2 \psi_1 & \lambda_2^2 \psi_2 & \lambda_3^2 \psi_3 & \lambda_4^2 \psi_4 & \lambda_5^2 \psi_5 & \lambda_6^2 \psi_6 \\ \lambda_1^2 \phi_1 & \lambda_2^2 \phi_2 & \lambda_3^2 \phi_3 & \lambda_4^2 \phi_4 & \lambda_5^2 \phi_5 & \lambda_6^2 \phi_6 \\ \lambda_1 \psi_1 & \lambda_2 \psi_2 & \lambda_3 \psi_3 & \lambda_4 \psi_4 & \lambda_5 \psi_5 & \lambda_6 \psi_6 \\ \lambda_1 \phi_1 & \lambda_2 \phi_2 & \lambda_3 \phi_3 & \lambda_4 \phi_4 & \lambda_5 \phi_5 & \lambda_6 \phi_6 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 \end{vmatrix} \quad (56b)$$

In addition, substitute (56) to the following equation

$$r[3] = -2 \frac{S_1^{21}}{\Delta}. \quad (57)$$

The equation (57) yields the solution for 3-soliton by obeying the following chosen

$$\psi_2 = -\phi_1^* \quad (58a)$$

$$\phi_4 = -\psi_3^* \quad (58b)$$

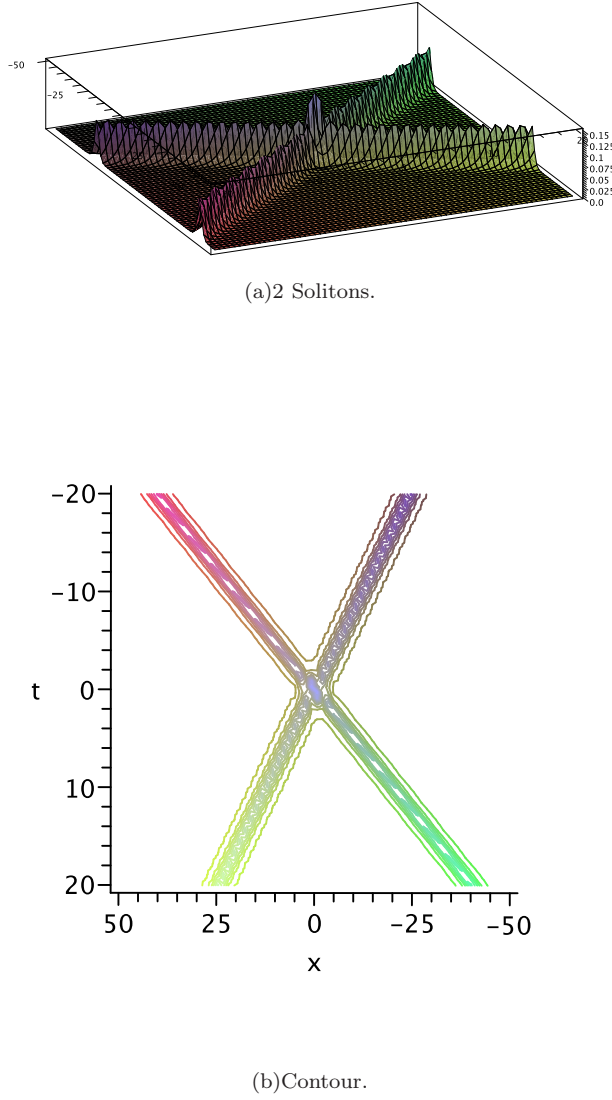


FIG. 1: 2 solitons.

$$\psi_6 = -\phi_5^* \quad (58c)$$

Below the illustration based on the above formulation for three solitons are given. By changing the parameters, several forms of multi-solitons can be obtained.

D. Conclusion

It has been shown that nonlinear refractive index in fiber optics materials, which has Kerr-like medium characteristics, will compensate dispersive effect of electromagnetic wave propagating in that fiber optics materials. The nonlinear evolution equation of electromagnetic wave due to the both effects can be solved by Darboux transformations. The advantage of Darboux transformation is that one can obtain multisolitons solution from trivial solution. This transformation has the similar form

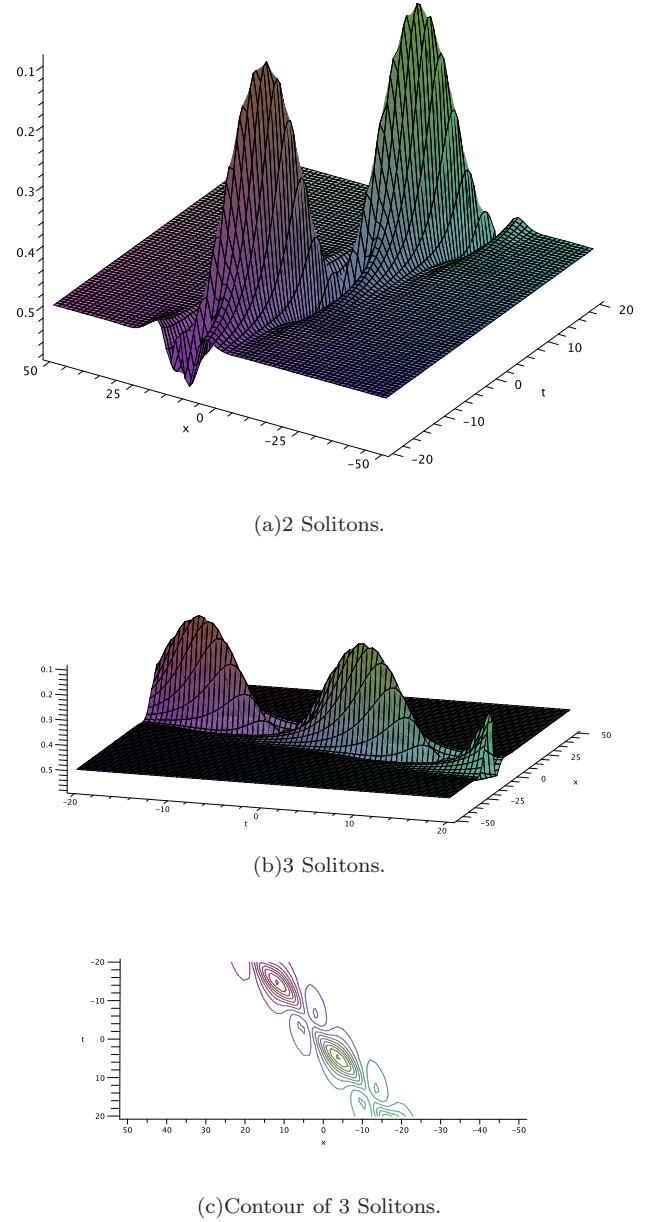


FIG. 2: 3 soliton contour

with the other methods which is used to solve nonlinear differential equations, i.e. Cole-Hopf transformations.

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- [2] V. B. Matveev, M. A. Salle, and A. V. Rybin, *Inverse Problems* **4**, 173 (1988), URL <http://stacks.iop.org/0266-5611/4/173>.